Weak Hamiltonian Finite Element Method for Optimal Control Problems

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A temporal finite element method based on a mixed form of Hamilton's weak principle is developed for dynamics and optimal control problems. The mixed form of Hamilton's weak principle contains both displacements and momenta as primary variables that are expanded in terms of nodal values and simple polynomial shape functions. Unlike other forms of Hamilton's principle, however, time derivatives of the momenta and displacements do not appear therein; instead, only the virtual momenta and virtual displacements are differentiated with respect to time. Based on the similarity in structure that is observed to exist between the mixed form of Hamilton's weak principle and variational principles governing classical optimal control problems, a temporal finite element formulation of the latter can be developed in a rather straightforward manner. Very crude shape functions (so simple that element numerical quadrature is not necessary) can be used to develop an efficient procedure for obtaining candidate solutions (i.e., those which satisfy all the necessary conditions) even for highly nonlinear problems. Solutions for some well-known problems in dynamics and optimal control are illustrated. The example dynamics problem involves a nonlinear time-marching problem. As optimal control examples, trajectory optimization problems with both fixed and open final time are treated.

Introduction

THIS paper examines a finite element approach to addressing optimal control problems. Hamilton's principle has traditionally been used in *analytical* mechanics as a method of obtaining the equations of motion for dynamical systems. Bailey,[1] followed by several others (see Refs. 2–4, for example), obtained direct solutions to dynamics problems using a form of Hamilton's principle known as the law of varying action, thus opening the door for its use in *computational* mechanics.

More recently, it has been shown that expression of Hamilton's law as a weak form (commonly referred to as Hamilton's weak principle or HWP) provides a powerful alternative to numerical solution of ordinary differential equations in the time domain.[5,6] The accuracy of the time-marching procedure derived in Refs. 5 and 6 is competitive with standard ordinary differential equation solvers. To derive an unconditionally stable algorithm in Ref. 5, however, reduced/selective element quadrature had to be used. Further computational advantages may be obtained in so-called mixed formulations of HWP in which the generalized coordinates and momenta appear as independent unknowns.[7] Therein, an unconditionally stable algorithm emerges for the linear oscillator with exact element quadrature. HWP also has been shown to be an ideal tool for obtaining periodic solutions for autonomous systems, as well as finding the corresponding transition matrix for perturbations about the periodic solution.[8] These are complex two-point boundary-value problems; its utility for these problems and its superior performance in mixed form strongly suggest that it could be used in optimal control problems.

In this paper we show that HWP in mixed form provides a useful parallel to optimal control problems. Although finite elements have been applied to optimal control problems,[9] the present formulation is believed to offer advantages over existing formulations. For example, it allows simple choices for shape functions in the finite element formulation,[7] which simplifies element quadrature.

We begin by summarizing HWP in both displacement and mixed forms. The mixed form is then applied to an initial value problem in dynamics. Then, making use of the well-known analogy between dynamics and optimal control, we develop our Hamiltonian weak form for optimal control problems. Finally, we apply it to two relatively simple optimal control problems to illustrate its power.

Hamilton's Weak Principle for Dynamics

In Refs. 1–8, one can see the potential of obtaining a direct solution in the time domain very much analogous to obtaining the solution of a beam deflection problem with the beam axial coordinate broken into several segements or finite elements. In the present case, however, it is the time interval that is broken into segments; thus, the term "finite elements in time" has been adopted by several investigators.

Only recently has a mixed formulation of HWP been investigated as a computational tool for finite elements in time.[7] In this section we will formulate the mixed form of HWP and illustrate its application to dynamics problems.

General Development

To this aim, let us consider an arbitrary holonomic mechanical system. The configuration is completely defined by a set of generalized coordinates q. Further, let us denote with $L(q, \dot{q}, t)$ the Lagrangian of the system, Q the set of nonconservative generalized forces applied to the system, and $p = \partial L / \partial \dot{q}$ the set of generalized momenta. The generalized coordinates q should be piecewise differentiable and the generalized momenta p will have discrete values at t_0 and t_f. (For a more mathematically rigorous discussion, see Ref. 10.) Then the following variational equation, known as HWP,[5] describes the real motion of the system between the two *known* times t_0

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and t_f:

$$\int_{t_0}^{t_f} \delta L \, dt + \int_{t_0}^{t_f} \delta q^T Q \, dt = \delta q_f^T \hat{p}_f - \delta q_0^T \hat{p}_0 \tag{1}$$

where δq, the variation of q, should be of the same class of functions as is q, $\delta q_f = \delta q(t = t_f)$ and $\delta q_0 = \delta q(t = t_0)$. Although HWP contains p in the form of discrete values at the end points denoted by the "hatted" quantities, this particular variational equation is said to be in displacement form because it only involves the variation of q. Endpoint conditions on the momenta, as derived from Eq. (1), can be shown to be of the natural or weak type.[5,6] Although this formulation has been shown to be of practical use in dynamics, an even more useful formulation may be derived if independent variations in both displacements and momenta are allowed, resulting in a mixed formulation.

To derive a mixed formulation in which q and p are independent field variables and treated the same at the endpoints, we introduce a Lagrange multiplier to enforce natural boundary conditions on q. Then, after defining the Hamiltonian as

$$H(q,p,t) \equiv p^T \dot{q} - L(q,\dot{q},t) \tag{2}$$

we take the variation of Eq. (2), substitute for δL in Eq. (1), and identify the Lagrange multiplier as p. After integration by parts to eliminate $\dot{q}$, the resulting variational equation is[7]

$$\int_{t_0}^{t_f} (\delta \dot{q}^T p - \delta \dot{p}^T q - \delta H + \delta q^T Q) \, dt$$
$$= \delta q_f^T \hat{p}_f - \delta q_0^T \hat{p}_0 - \delta p_f^T \hat{q}_f + \delta p_0^T \hat{q}_0 \tag{3}$$

This is called a mixed formulation because it contains independent variations of q and p. It is also in the "weakest" possible form in the sense that all boundary conditions are of the natural type, enforced by the variational equation for unconstrained variations. Note that now p and q should have discrete values at t_0 and t_f (again denoted by hats), p and q should be piecewise continuous, and δp and δq should be continuous and piecewise differentiable (C^0).

There are two main advantages of the mixed formulation over the displacement formulation. The first advantage is that the mixed formulation generally provides a more accurate solution for a given level of computational effort than does the displacement formulation. The second advantage is that a simpler choice of shape functions is allowed. Note in Eq. (3) that time derivatives of δq and δp are present. However, no time derivatives of q and p exist. Therefore, it is possible to implement linear shape functions for δq and δp and constant functions for q and p within each element.

Let us break the time interval from t_0 to t_f into N elements. The nodal values of these elements are t_i for $i = 1, \ldots, N+1$ where $t_0 = t_1$ and $t_f = t_{N+1}$. We define a nondimensional elemental time τ as

$$\tau = \frac{t - t_i}{t_{i+1} - t_i} = \frac{t - t_i}{\Delta t_i} \tag{4}$$

The linear shape functions for the virtual coordinates and momenta are

$$\delta q = \delta q_i (1 - \tau) + \delta q_{i+1} \tau \tag{5a}$$

$$\delta p = \delta p_i (1 - \tau) + \delta p_{i+1} \tau \tag{5b}$$

For the generalized coordinates and momenta

$$q = \begin{cases} \hat{q}_i & \text{if } \tau = 0 \\ \bar{q}_i & \text{if } 0 < \tau < 1 \\ \hat{q}_{i+1} & \text{if } \tau = 1 \end{cases} \tag{6}$$

and

$$p = \begin{cases} \hat{p}_i & \text{if } \tau = 0 \\ \bar{p}_i & \text{if } 0 < \tau < 1 \\ \hat{p}_{i+1} & \text{if } \tau = 1 \end{cases} \tag{7}$$

It is important to understand that the equalities $\hat{q}_1 = q(t_0)$, $\hat{p}_1 = p(t_0)$, $\hat{q}_{N+1} = q(t_f)$, and $\hat{p}_{N+1} = p(t_f)$ are enforced as natural (weak) boundary conditions. In other words, the hatted values of q and p at the beginning and end of our time marching scheme are the discrete values of q and p that are needed in the mixed formulation. When these shape functions are substituted into Eq. (3), one can either generate an implicit time-marching procedure for nonlinear problems or apply standard finite element assembly procedures to solve periodic or two-point boundary value problems.[8] When this formulation is applied to the linear oscillator, a time-marching algorithm emerges that is unconditionally stable. Higher-order (so-called p-version) elements could be developed, and they would certainly be attractive for linear problems or for nonlinear problems with nonlinearities of low order. For nonlinear problems in general, use of the crude shape functions allowable with the mixed method would seem to be more efficient than use of higher-order shape functions in a p-version. The reason for this is that, with an exception of the term involving Q, which may contain time explicitly, all element quadrature can be done by inspection, regardless of the order of the nonlinearities. Detailed comparison of these methods is beyond the scope of the present paper but is being undertaken by the first author at this time.

Example 1: Nonlinear Initial-Value Problems

Applying the shape functions of Eqs. (5–7) to Eq. (3) for an initial-value problem, we obtain a recursive set of nonlinear algebraic equations of the form

$$f_j(\bar{q}_i, \hat{q}_{i+1}, \bar{p}_i, \hat{p}_{i+1}) = 0 \qquad j = 1, 2, \ldots, n \tag{8}$$

where n is four times the number of degrees of freedom of the system. Equation (8) can be solved by a Newton-Raphson

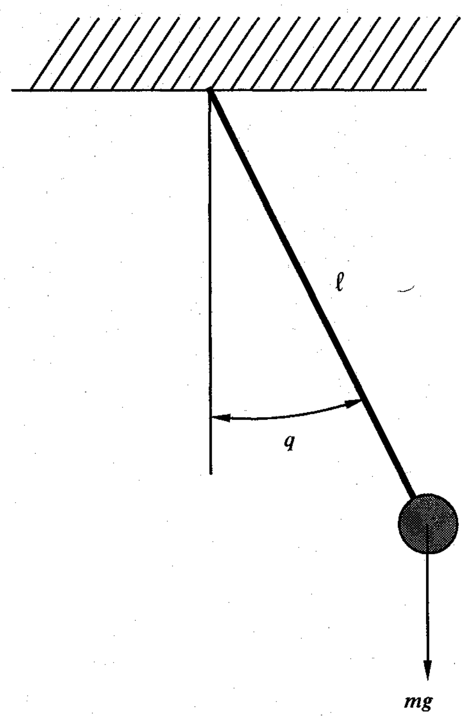

Fig. 1 Nomenclature for example 1, a simple pendulum composed of a lumped mass m and a weightless bar of length l.

method yielding an implicit time-marching procedure. The key advantage of using finite elements and a weak variational approach over numerical integration is that the solution (for linear problems) is stable for *all* time steps. In other words, no matter how large a time step is used, a finite approximation of the solution will be obtained. This unconditional stability is obtained without ad hoc procedures such as selective or reduced element quadrature, which are necessary in displacement formulations.

We also point out that

$$2\bar{q}_i = \hat{q}_i + \hat{q}_{i+1} \tag{9a}$$

$$2\bar{p}_i = \hat{p}_i + \hat{p}_{i+1} \tag{9b}$$

Thus, it is possible to cut the number of equations and unknowns in half. This can be very useful for a multi-degree-of-freedom problem.

Consider a simple pendulum composed of a lumped mass m and a weightless bar of length l (see Fig. 1). The single generalized coordinate q is the angular displacement of the bar from the vertical. Denoting the kinetic energy of the system with K and the potential energy with V, then we may define the following:

$$K = \tfrac{1}{2} m l^2 \dot{q}^2 \tag{10a}$$

$$V = mgl(1 - \cos q) \tag{10b}$$

$$L = K - V \tag{10c}$$

$$p = \frac{\partial L}{\partial \dot{q}} = ml^2 \dot{q} \tag{10d}$$

$$H = \frac{p^2}{2ml^2} + mgl(1 - \cos q) \tag{10e}$$

There are no nonconservative forces Q applied to this system.

Substituting $t = t_i + \tau \Delta t_i$, along with Eq. (10), and substituting the shape functions defined in Eqs. (5–7) into Eq. (3) we obtain

$$\begin{aligned}\Delta t_i \int_0^1 \Bigg\{ &\left(\frac{\delta q_{i+1} - \delta q_i}{\Delta t_i}\right)\bar{p}_i - mgl \sin\bar{q}_i[\delta q_i(1-\tau) + \delta q_{i+1}\tau] \\ &- \left(\frac{\delta p_{i+1} - \delta p_i}{\Delta t_i}\right)\bar{q}_i - \left(\frac{\bar{p}_i}{ml_2}\right)[\delta p_i(1-\tau) + \delta p_{i+1}\tau]\Bigg\} d\tau \\ &- \delta q_{i+1}\hat{p}_{i+1} + \delta p_{i+1}\hat{q}_{i+1} + \delta q_i \hat{p}_i - \delta p_i \hat{q}_i = 0\end{aligned} \tag{11}$$

Carrying out the integration by inspection, we obtain the four independent equations

$$\hat{p}_i - \bar{p}_i - \frac{mgl\,\Delta t_i \sin\bar{q}_i}{2} = 0 \tag{12a}$$

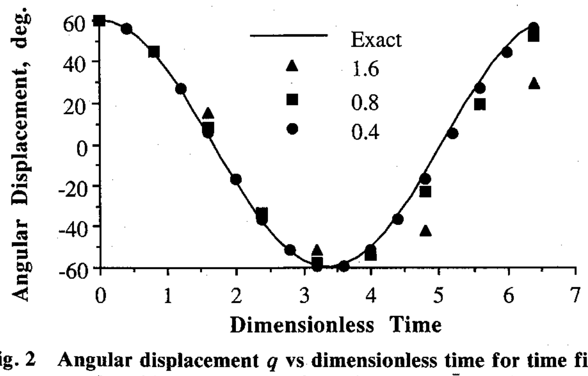

Fig. 2 Angular displacement q vs dimensionless time for time finite element results with three values of the time step $\Delta\bar{t}$ and for the exact elliptic integral solution.

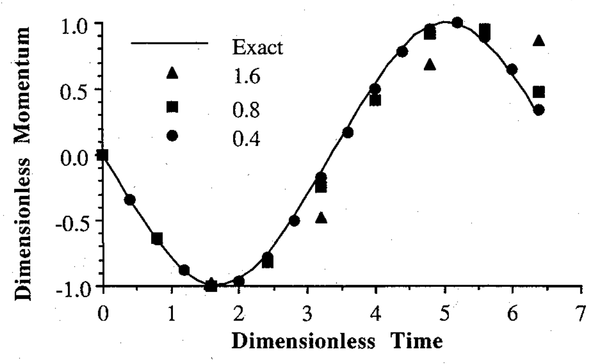

Fig. 3 Dimensionless momentum $p/ml^2\omega$ vs dimensionless time for time finite element results with three values of the time step $\Delta\bar{t}$ and for the exact elliptic integral solution.

$$\bar{p}_i - \hat{p}_{i+1} - \frac{mgl\,\Delta t_i \sin\bar{q}_i}{2} = 0 \tag{12b}$$

$$\bar{q}_i - \hat{q}_i - \frac{\bar{p}_i\,\Delta t_i}{2ml^2} = 0 \tag{12c}$$

$$\hat{q}_{i+1} - \bar{q}_i - \frac{\bar{p}_i\,\Delta t_i}{2ml^2} = 0 \tag{12d}$$

There are six unknowns; however, for an initial-value problem, we will specify $\hat{q}_i$ and $\hat{p}_i$ and solve for the remaining unknowns as outlined below. Thus, Eq. (12) is of the form of Eq. (8).

Recall that i ranges from 1 to $N+1$. We start with $i = 1$ and specify $\hat{q}_1$ and $\hat{p}_1$ (i.e., the initial conditions) and solve for $\bar{q}_1, \bar{p}_1, \hat{q}_2$, and $\hat{p}_2$. Then, we let $i = 2$ and specify $\hat{q}_2$ and $\hat{p}_2$ (we just found those values) and solve for the new four unknowns. We repeat this process until $i = N + 1$.

For this simple pendulum example, we will nondimensionalize the variables as follows. If we define $\omega^2 = g/l$, then a dimensionless time step $\Delta\bar{t}$ may be defined that does not vary with i so that $\Delta\bar{t} = \omega\,\Delta t_i$. Also, instead of solving directly for p, we will solve for the dimensionless $p/ml^2\omega$.

We will start our pendulum at $\hat{q}_1 = 60$ deg and $\hat{p}_1 = 0.0$. The equations will be solved for $\Delta\bar{t} = 0.4$, 0.8, and 1.6. Graphs of the solutions are shown in Figs. 2 and 3 and compared to the exact elliptic integral solution.[11] (Since the element values are simply the average of the element nodal values, only the nodal values are given for all the results in this paper.) From Figs. 2 and 3, it is easily seen that $\Delta\bar{t} = 0.4$ gives acceptable results for both displacement and angular velocity. Also, note that even the large 1.6 time step yields a finite approximation of the exact solution.

Weak Principle for Optimal Control

A definite analogy[12] exists between the mixed formulation of HWP in dynamics and the first variation of the performance index in optimal control theory. Specifically, there is a similarity between the generalized coordinates and generalized momenta in dynamics and the states and costates in optimal control theory.

Since the mixed formulation has proved to be so valuable in dynamics, we will derive a weak form based on the variation of the performance index. When deriving this formulation, we will keep two things in mind. First, the resulting formulation must satisfy the Euler-Lagrange equations and boundary conditions that have already been established in optimal control theory.[12] Second, all strong boundary conditions will be transformed into natural or weak boundary conditions. After development of the weak form and finite element discretization, we will present two example problems.

General Development

It is desired to develop a solution strategy for optimal control problems based on finite elements in time. In an attempt to make the solution scheme as general as possible, all strong boundary conditions will be transformed into natural boundary conditions. This is done so that the shape functions used for the test functions can be chosen from a less restrictive class of functions. For example, if there is a strong boundary condition on one of the states at the initial time (i.e., an initial condition) then the shape function chosen for the variation of that state must equal zero at the initial time.[13] It would be advantageous if one could choose the same shape functions for every optimal control problem. This is possible if there are no strong boundary conditions that must be satisfied by the shape functions.

The idea of transforming strong boundary conditions to natural boundary conditions[14] revolves around adjoining a constraint equation to the performance index with an unknown Lagrange multiplier. The variation of the performance index is then taken in a straightforward manner. Through appropriate integration by parts, one may show that the Euler-Lagrange equations are identical to those derived in classical textbooks[12] and that the boundary conditions are the same, only stated weakly instead of strongly.

Consider a system defined by a set of n states x and a set of m controls u. Furthermore, let the system be governed by a set of state equations of the form $\dot{x}=f(x,u,t)$. In this paper, we will restrict our attention to only that class of problems where x, u, and f are continuous. We may denote elements of the performance index J_0 with an integrand $L(x,u,t)$ and discrete functions of the states and time $\phi[x(t),t]$ defined only at the initial and final times t_0 and t_f. In addition, any constraints imposed on the states and time at the initial and final times may be placed in sets of functions $\psi[x(t),t]$. These constraints may be adjoined to the performance index by discrete Lagrange multipliers ν defined at t_0 and t_f. Finally, we may adjoin the state equations to the performance index with a set of Lagrange multiplier functions $\lambda(t)$, which will be referred to as costates. For variable t_f, this yields a performance index of the form

$$J_0=\int_{t_0}^{t_f}\left[L(x,u,t)+\lambda^T(f-\dot{x})\right]\mathrm{d}t+\Phi|_{t_0}^{t_f} \tag{13}$$

where $\Phi = \phi[x(t),t]+\nu^T\psi[x(t),t]$. The constraints to be adjoined to J_0 above are simply that the states be continuous at the initial and final times. Introducing

$$x|_{t_0}\triangleq\lim_{t\to t_0^+}x(t) \quad \text{and} \quad x|_{t_f}\triangleq\lim_{t\to t_f^-}x(t) \tag{14}$$

$$\hat{x}|_{t_0}\triangleq x(t_0) \quad \text{and} \quad \hat{x}|_{t_f}\triangleq x(t_f) \tag{15}$$

we weakly enforce continuity by adjoining $\alpha^T(x-\hat{x})|_{t_0}^{t_f}$ to J_0 where α is a set of discrete unknown Lagrange multipliers defined only at t_0 and t_f. The new performance index is

$$J=\int_{t_0}^{t_f}\left[L(x,u,t)+\lambda^T(f-\dot{x})\right]\mathrm{d}t+\Phi|_{t_0}^{t_f}+\alpha^T(x-\hat{x})|_{t_0}^{t_f} \tag{16}$$

To derive the weak principle, it is necessary to determine δJ, the first variation of J. Denoting with $\delta x(t_f)$ and $\delta\lambda(t_f)$ the variations of x and λ at $t=t_f$ when holding t_f fixed, and letting $\Delta x(t_f)$ and $\Delta\lambda(t_f)$ be the variations of x and λ at $t=t_f$ when t_f is allowed to be free, we can express the variations at $t=t_f$ by the linear equations[15]

$$\delta x(t_f)=\Delta x(t_f)-\dot{x}|_{t_f}\delta t_f \quad \text{and} \quad \delta\lambda(t_f)=\Delta\lambda(t_f)-\dot{\lambda}|_{t_f}\delta t_f \tag{17}$$

The first variation of J is

$$\begin{aligned}\delta J=\int_{t_0}^{t_f}&\left\{\delta\lambda^T(f-\dot{x})-\delta\dot{x}^T\lambda+\delta x^T\left[\left(\frac{\partial L}{\partial x}\right)^T+\left(\frac{\partial f}{\partial x}\right)^T\lambda\right]\right.\\&\left.+\delta u^T\left[\left(\frac{\partial L}{\partial u}\right)^T+\left(\frac{\partial f}{\partial u}\right)^T\lambda\right]\right\}\mathrm{d}t\\&+\delta t_f\left[L+\lambda^T(f-\dot{x})+\frac{\partial\Phi}{\partial t}\right]\Bigg|_{t_f}+\delta\nu^T\psi|_{t_0}^{t_f}\\&+\Delta x^T\left(\frac{\partial\Phi}{\partial x}\right)\Bigg|_{t_0}^{t_f}+\delta\alpha^T(x-\hat{x})|_{t_0}^{t_f}+\alpha^T(\Delta x-\Delta\hat{x})|_{t_0}^{t_f}\end{aligned} \tag{18}$$

A necessary condition for an extremal of J is that the first variation be zero. Also, the admissible variations of the states must be continuous at the initial and final times and therefore $(\Delta x-\Delta\hat{x})|_{t_0}^{t_f}=0$. For notational convenience we will define

$$\hat{\lambda}|_{t_0}=\frac{\partial\Phi}{\partial x}\bigg|_{t_0} \quad \text{and} \quad \hat{\lambda}|_{t_f}=\frac{\partial\Phi}{\partial x}\bigg|_{t_f} \tag{19}$$

Finally, to ensure that none of the necessary conditions have been altered, the $\delta\dot{x}$ term will be integrated by parts. This results in

$$\begin{aligned}\int_{t_0}^{t_f}&\left\{\delta\lambda^T(f-\dot{x})+\delta x^T\left[\dot{\lambda}+\left(\frac{\partial L}{\partial x}\right)^T+\left(\frac{\partial f}{\partial x}\right)^T\lambda\right]\right.\\&\left.+\delta u^T\left[\left(\frac{\partial L}{\partial u}\right)^T+\left(\frac{\partial f}{\partial u}\right)^T\lambda\right]\right\}\mathrm{d}t\\&+\delta t_f\left[L+\lambda^T(f-\dot{x})+\frac{\partial\Phi}{\partial t}\right]\Bigg|_{t_f}+\delta\nu^T\psi|_{t_0}^{t_f}\\&+\Delta x^T\hat{\lambda}|_{t_0}^{t_f}+\delta\alpha^T(x-\hat{x})|_{t_0}^{t_f}-\delta x^T\lambda|_{t_0}^{t_f}=0\end{aligned} \tag{20}$$

Using Eq. (17) and noting that $\delta x|_{t_0}=\Delta x|_{t_0}$ since t_0 is fixed, then the preceding equation can be simplified to the following:

$$\begin{aligned}\int_{t_0}^{t_f}&\left\{\delta\lambda^T(f-\dot{x})+\delta x^T\left[\dot{\lambda}+\left(\frac{\partial L}{\partial x}\right)^T+\left(\frac{\partial f}{\partial x}\right)^T\lambda\right]\right.\\&\left.+\delta u^T\left[\left(\frac{\partial L}{\partial u}\right)^T+\left(\frac{\partial f}{\partial u}\right)^T\lambda\right]\right\}\mathrm{d}t\\&+\delta t_f\left[L+\lambda^Tf+\frac{\partial\Phi}{\partial t}\right]\Bigg|_{t_f}+\delta\nu^T\psi|_{t_0}^{t_f}\\&+\Delta x^T(\hat{\lambda}-\lambda)|_{t_0}^{t_f}+\delta\alpha^T(x-\hat{x})|_{t_0}^{t_f}=0\end{aligned} \tag{21}$$

We will now compare the aforementioned Euler-Lagrange equations with the well-known optimal control equations presented in Ref. 12. The coefficients of $\delta\lambda^T$, δx^T, and δu^T in the integrand, when set equal to zero, correspond to Eqs. (2.8.15–2.8.17) from Ref. 12. There are also four trailing terms in Eq. (21) from which the boundary conditions of Ref. 12 can be determined. Namely, the requirement for the coefficient of δt_f to vanish is equivalent to Eq. (2.8.20). The requirement for the coefficient of $\delta\nu^T$ to vanish at $t=t_f$ yields Eq. (2.8.21). The requirement for the coefficient of Δx^T to vanish at $t=t_f$ shows that the value of $\lambda|_{t_f}$ equals $\hat{\lambda}|_{t_f}$ as given in Eq. (19), which corresponds to Eq. (2.8.19). Finally, the requirement for the coefficient of $\delta\alpha^T$ to vanish at $t=t_0$ requires the value of $x|_{t_0}$ equal $\hat{x}|_{t_0}$, in accordance with Eq. (2.8.18).

Three additional boundary conditions are present in the above formulation. One is the requirement for the coefficient of $\delta\alpha^T$ vanish at $t=t_f$, which demands that the value of $x|_{t_f}$ equal $\hat{x}|_{t_f}$. The second is the requirement for the coefficient of Δx^T to vanish at $t=t_0$, which demands that the value of $\lambda|_{t_0}$ equal $\hat{\lambda}|_{t_0}$ as given in Eq. (19). These two conditions enforce continuity of the states at the final time and continuity of the costates at the initial time. The third and last boundary condition is the requirement for the coefficient of $\delta\nu^T$ at $t=t_0$ to vanish, which demands that $\psi[x(t_0),t_0]=0$. Again, all boundary conditions were cast in the form of natural boundary conditions so that the shape functions chosen for δx and $\delta\lambda$ will not have to satisfy any particular boundary conditions.

Having satisfied the requirement that none of the fundamental equations are altered, we may now derive the weak formulation from Eq. (21). Recall from the mixed formulation for dynamics that no time derivatives of q or p appear in the formulation. Similarly, the time derivatives of x and λ are not to appear in the present weak formulation for optimal control. By noting that α is a Lagrange multiplier whose only restriction is that $\delta\alpha$ be independent of Δx, $\delta\nu$, and δt_f and that α has the units of the costates, we then choose $\delta\alpha=\Delta\lambda$. (Note that there is no unique choice for $\delta\alpha$, but this one will lead to a successful solution strategy.) The Δx and $\Delta\lambda$ terms can now be eliminated from Eq. (21) using Eq. (17), and after integration by parts the weak form becomes

$$\int_{t_0}^{t_f}\left\{\delta\dot{\lambda}^T x-\delta\dot{x}^T\lambda+\delta x^T\left[\left(\frac{\partial L}{\partial x}\right)^T+\left(\frac{\partial f}{\partial x}\right)^T\lambda\right]+\delta\lambda^T f+\delta u^T\left[\left(\frac{\partial L}{\partial u}\right)^T+\left(\frac{\partial f}{\partial u}\right)^T\lambda\right]\right\}\mathrm{d}t$$
$$+\delta t_f\left[L+\lambda^T f+\frac{\partial\Phi}{\partial t}\right]\Bigg|_{t_f}+\delta\nu^T\psi\big|_{t_0}^{t_f}$$
$$+\delta x^T\hat{\lambda}\big|_{t_0}^{t_f}-\delta\lambda^T\hat{x}\big|_{t_0}^{t_f}+(\hat{\lambda}-\lambda)^T\dot{x}\big|_{t_f}\delta t_f$$
$$-(\hat{x}-x)^T\dot{\lambda}\big|_{t_f}\delta t_f=0 \tag{22}$$

After noting that the last two terms are equal to zero in accordance with the natural boundary conditions, one can discard those terms without changing the necessary conditions. Also, we note that for most problems, the initial conditions are given for all n states and thus, in accordance with Eq. (19), all the initial costates are unknown. Therefore, instead of treating elements of ν at $t=t_0$ as unknowns and replacing $\hat{\lambda}|_{t_0}$ with these unknowns, we will instead treat $\hat{\lambda}|_{t_0}$ as unknowns and eliminate the $\delta\nu|_{t_0}$ equations from the weak principle. We hasten to point out that the elements of $\hat{x}|_{t_0}$ are the initial conditions. The final form of the weak principle is then given as

$$\int_{t_0}^{t_f}\left\{\delta\dot{\lambda}^T x-\delta\dot{x}^T\lambda+\delta x^T\left[\left(\frac{\partial L}{\partial x}\right)^T+\left(\frac{\partial f}{\partial x}\right)^T\lambda\right]+\delta\lambda^T f+\delta u^T\left[\left(\frac{\partial L}{\partial u}\right)^T+\left(\frac{\partial f}{\partial u}\right)^T\lambda\right]\right\}\mathrm{d}t$$
$$+\delta t_f\left(L+\lambda^T f+\frac{\partial\Phi}{\partial t}\right)\Bigg|_{t_f}$$
$$+\delta\nu^T\psi\big|_{t_f}+\delta x^T\hat{\lambda}\big|_{t_0}^{t_f}-\delta\lambda^T\hat{x}\big|_{t_0}^{t_f}=0 \tag{23}$$

This is the governing equation for the weak Hamiltonian method for optimal control problems of the form specified. It will serve as the basis for the finite element discretization described below for constructing candidate solutions (i.e., solutions that satisfy all the necessary conditions). It should be noted that normally one will encounter various types of inequality constraints on the states and controls, as well as discontinuities in the states and state equations in problems that deal with optimal control. These aspects will be treated in future papers.

Finite Element Discretization

As in dynamics, we may choose linear shape functions for δx and $\delta\lambda$. We may choose piecewise constant shape functions for x and λ. Thus, we will be working with shape functions similar to those of Eqs. (5-7). In addition, note that the time derivatives of u and δu do not appear in the formulation. Thus, we let

$$u=\bar{u}_i \tag{24a}$$

$$\delta u=\delta\bar{u}_i \tag{24b}$$

Plugging in the shape functions described for x, λ, and u, substituting $t=t_i+\tau\Delta t_i$, and carrying out the element quadrature over τ from 0 to 1, we obtain a general algebraic form of our Hamiltonian weak form for optimal control problems of the form specified. Note that if the time t does not appear explicitly in the problem formulation then all integration is exact and can be done by inspection. If t does appear explicitly, then t may be approximated by a constant value over each element (as are x, λ, and u) and the integration may still be done by inspection. We further note that for this two-point boundary-value problem we must assemble our elements over the entire time interval. Only the nodal values (the hatted quantities) of the states and costates at the initial and final times appear in the algebraic equations.

After integration and assembly, Eq. (23) becomes a system of algebraic equations. In fact, for N elements, there are $2n(N+1)+mN+q+1$ equations and $2n(N+2)+mN+q+1$ unknowns. Therefore, $2n$ of the $4n$ endpoint values for the states and costates ($\hat{x}_0$, $\hat{\lambda}_0$, $\hat{x}_f$, and $\hat{\lambda}_f$) must be specified. In general, $\hat{x}_0$ (the initial conditions) is known in accordance with physical constraints. Also, $\hat{\lambda}_f$ can be specified in terms of other unknowns with the use of Eq. (19). Now we have the same number of equations as unknowns. These equations may be used for any optimal control problem of the form specified. One simply needs to substitute the appropriate f, L, ϕ, ψ, and boundary conditions into Eq. (23) for a given problem.

Normally, Eq. (23) can be solved by expressing the Jacobian explicitly and using a Newton-Raphson solution procedure. For the example problems that follow, the iteration procedure will converge quickly for a small number of elements with a trivial initial guess. Then, the answers obtained for a small number of elements can be used to generate initial guesses for a higher number of elements. Thus, a large number of elements can be solved with a very efficient runtime on the computer.

Unfortunately, for some highly nonlinear problems, trivial initial guesses may not be adequate. This problem may be overcome by noting that the δx_i and δx_{i+1} equations in Eq. (23) are $n(N+1)$ equations that happen to be linear in the $n(N+1)$ unknown costates. Now, we need $n(N+1)$ fewer initial guesses and we can solve for the costates symbolically in terms of the other unknowns if n is not too large. This can be useful since generating appropriate initial estimates for the values of the costates may not be straightforward.

Although the nodal values $\hat{x}_i$ and $\hat{\lambda}_i$ for $2<i<N$ (on the interior of the time interval) do not appear in the algebraic equations, their values can be easily recovered after the solution, since the element values ($\bar{x}_i$ and $\bar{\lambda}_i$) are just the mean of the nodal values [see Eq. (9)]. Although the shape function for the control u only defines a constant value within the element, values of u at additional points are available. For instance, once the nodal values for the states and costates are found, then one may use the optimality condition ($\partial H/\partial u=0$) to solve for u at a nodal point. Also, if the states and costates are approximated by some continuous curve fit through the nodal values obtained from the solution, then the control could be approximated at any instant in time by using the optimality condition.

Example 2: Fixed-Final-Time Problem

As the first optimal control problem, we will examine the transfer of a particle to a rectilinear path (see Fig. 4). This is an example taken from Ref. 12, article 2.4. We will let $x_{(1)}$ and $x_{(2)}$ denote the position of the particle at a given time and $x_{(3)}$ and $x_{(4)}$ denote the particle's velocity at a given time. (A

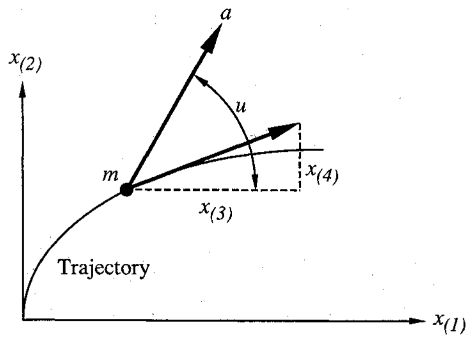

Fig. 4 Nomenclature for example 2, the transfer of a particle to a rectilinear path.

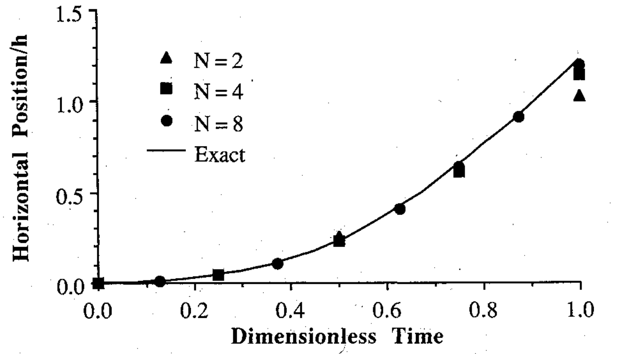

Fig. 5 Dimensionless horizontal position $x_{(1)}/h$ vs t/T. (Note that the final horizontal component of position is not specified.)

subscripted number in parentheses refers to the state index to avoid confusion with the element index.) The thrust angle u is the control and the particle has mass m and a constant acceleration a.

The state equations are defined as

$$\dot{x}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}x+\begin{Bmatrix}0\\0\\a\cos u\\a\sin u\end{Bmatrix}=f \tag{25}$$

The final time T is fixed and we would like to maximize the final horizontal component of velocity. Thus,

$$L=0 \tag{26a}$$

$$\phi=\lfloor 0\quad 0\quad 1\quad 0\rfloor\hat{x}_f \tag{26b}$$

There are also two terminal constraints on the states. These are that the particle arrive with a fixed final height (h) and that the final vertical component of velocity be zero. We do not care what the final horizontal component of position is. These constraints can be stated analytically as

$$\psi=\begin{bmatrix}0&1&0&0\\0&0&0&1\end{bmatrix}\hat{x}_f-\begin{Bmatrix}h\\0\end{Bmatrix} \tag{27}$$

The initial conditions are $x(0)=\hat{x}_0=\lfloor 0\quad 0\quad 0\quad 0\rfloor^T$. Finally, we will eliminate the unknown $\hat{\lambda}_f$ by writing it in terms of other unknowns. In accordance with Eq. (19),

$$\hat{\lambda}_f=\begin{Bmatrix}0\\v_1\\1\\v_2\end{Bmatrix} \tag{28}$$

We have now defined the f, L, ϕ, ψ, and boundary conditions to substitute into Eq. (23).

With $L=0$, and t_f fixed, along with all zero initial conditions on the states, then the general formulation of Eq. (23) takes the following algebraic form:

$$\begin{aligned}\sum_{i=1}^{N}\Bigg\{&\delta x_i^T\left[-\bar{\lambda}_i-\frac{\Delta t_i}{2}\left(\frac{\partial\bar{f}_i}{\partial\bar{x}_i}\right)^T\bar{\lambda}_i\right]+\delta\lambda_i^T\left(\bar{x}_i-\frac{\Delta t_i}{2}\bar{f}_i\right)\\&+\delta x_{i+1}^T\left[\bar{\lambda}_i-\frac{\Delta t_i}{2}\left(\frac{\partial\bar{f}_i}{\partial\bar{x}_i}\right)^T\bar{\lambda}_i\right]-\delta\lambda_{i+1}^T\left(\bar{x}_i+\frac{\Delta t_i}{2}\bar{f}_i\right)\\&-\delta u_i^T\,\Delta t_i\left(\frac{\partial\bar{f}_i}{\partial\bar{u}_i}\right)^T\bar{\lambda}_i\Bigg\}-\delta v^T\psi\\&+\delta x_1^T\hat{\lambda}_1-\delta x_{N+1}^T\hat{\lambda}_{N+1}+\delta\lambda_{N+1}^T\hat{x}_{N+1}=0\end{aligned} \tag{29}$$

where $\bar{f}_i=f(x=\bar{x}_i,u=\bar{u}_i)$. This produces a system of nonlinear algebraic equations whose size depends on the number of elements N.

These equations are solved by choosing $\Delta t_i=\Delta t=t_f/N$ for all i, expressing the Jacobian explicitly and using a Newton-Raphson algorithm. For $N=2$, suitable initial guesses for the nonlinear iterative procedure can be found by simply choosing element values that are not too different from the boundary conditions. The results from solving the $N=2$ equations are then used to obtain the initial guesses for arbitrary N by simple interpolation. In all results obtained to date for this problem, no additional steps are necessary to obtain results as accurate as desired.

Representative numerical results for all four states vs dimensionless time t/T are presented in Figs. 5–8. For this example, we have taken $h=100$, $T=20$, and $4h/aT^2=0.8897018$. (This last number is chosen to yield a value of 75 deg for the initial control angle of the exact solution available in Ref. 12.) The results for two, four, and eight elements are plotted against the exact solution. It can easily be seen that $N=8$ gives acceptable results for all the states. In Fig. 9, the control angle u vs dimensionless time t/T is presented. Once again, the results are seen to be excellent for $N=8$.

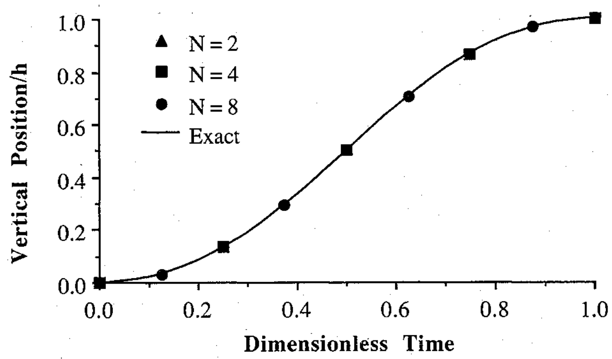

Fig. 6 Dimensionless vertical position $x_{(2)}/h$ vs t/T. (The final height is constrained to be h at the final time.)

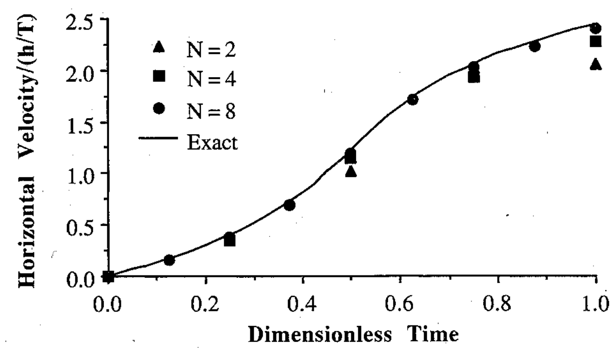

Fig. 7 Dimensionless horizontal velocity $x_{(3)}T/h$ vs t/T. [Note that the performance index $J=x_{(3)}(T)$.]

Three of the four costates are constants for all time and this method yields two of these exactly. The third costate is very close to the exact answer. The fourth costate corresponding to the vertical component of velocity $\lambda_{(4)}$ is shown in Fig. 10. The results compare nicely with the exact results.

A plot of the relative error of the performance index $J = \hat{x}_{f,(3)}$ and the endpoint multiplier ν_1 vs the number of elements is shown in Fig. 11. It is seen to be nearly a straight line on a log-log scale. The slope of the line is about -2, which indicates that the error varies inversely with the square of N, similar to a-posteriori error bounds, as formulated in usual finite element applications.[10] Notice in Fig. 11 that there is a bend in the endpoint multiplier curve. It is not unusual for mixed formulations to have an error curve that is not monotonically decreasing. It should be noted that developments of mathematical proofs for convergence and expressions for error bounds are not state-of-the-art for mixed methods.

Example 3: Free-Final-Time Problem

The second optimal control problem is similar to example 2, except that now the final time is free and we would like to obtain a given horizontal component of velocity (U) in the minimum time (see problem 9, article 2.7 of Ref. 12). Our algorithm from the preceding example is readily modified to fit this problem by noting the following changes. The performance index is now the final time T; so $\phi = 0$ and $L = 1$. Also, there is an additional endpoint constraint on the states, namely, that $x_{(3)} = U$. With these changes, we have

$$\hat{\lambda}_f = \begin{Bmatrix} 0 \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{Bmatrix} \tag{30}$$

$$\psi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \hat{x}_f - \begin{Bmatrix} h \\ U \\ 0 \end{Bmatrix} \tag{31}$$

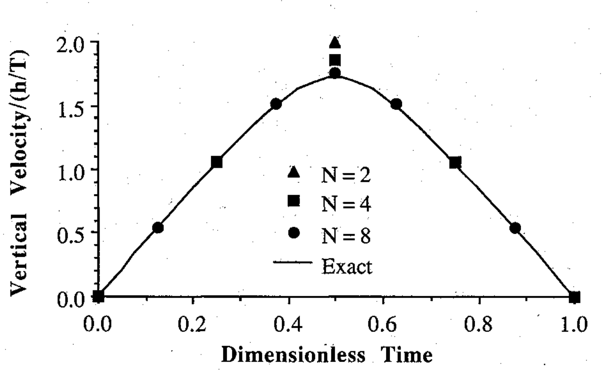

Fig. 8 Dimensionless vertical velocity $x_{(4)}T/h$ vs t/T. (The final vertical component of velocity is constrained to be zero at the final time.)

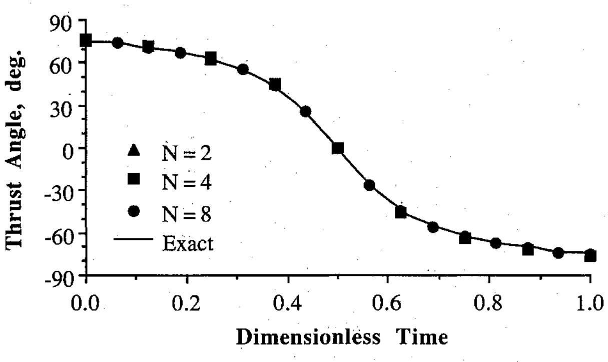

Fig. 9 Control angle u vs t/T.

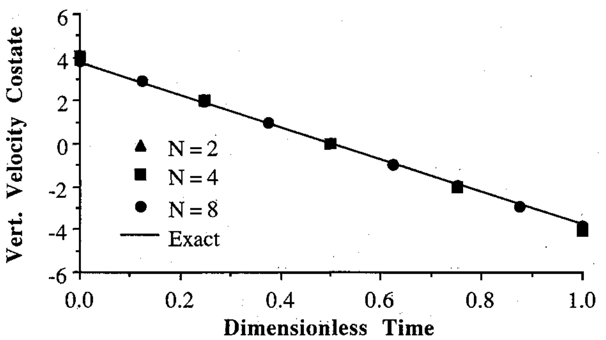

Fig. 10 Vertical velocity costate $\lambda_{(4)}$ vs t/T. (The results for this costate are the least accurate of all the costates.)

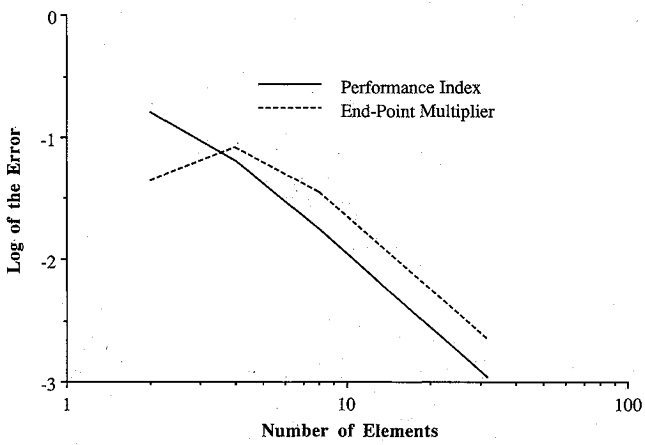

Fig. 11 Relative error of the performance index $x_{(3)}(T)$ and the endpoint multiplier ν_1 vs N.

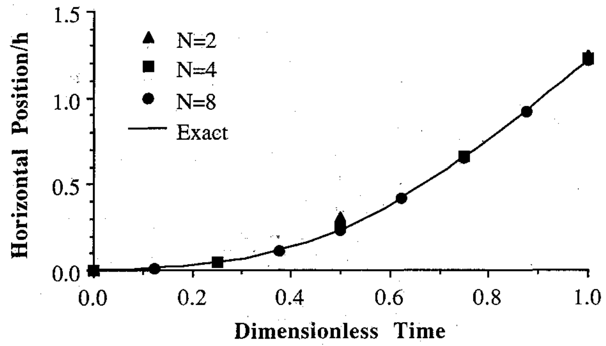

Fig. 12 Dimensionless horizontal position $x_{(1)}/h$ vs t/T. (Note that the final horizontal component of position is not specified.)

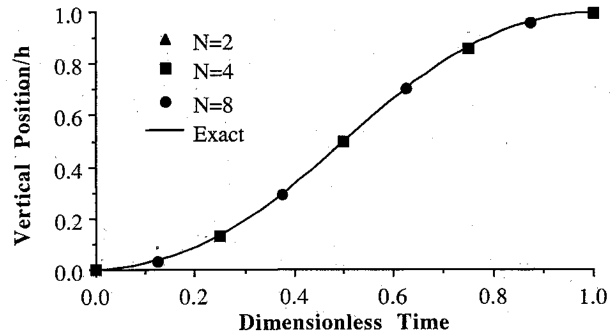

Fig. 13 Dimensionless vertical position $x_{(2)}/h$ vs t/T. (The final height is constrained to be h at the final time.)

Along with these changes to our equations, we also pick up the additional δt_f equation. The new system of equations is solved in the same manner described previously. Again, trivial initial guesses are satisfactory for $N = 2$, and these answers are used to obtain initial guesses for arbitrary N.

Representative numerical results for all four states vs dimensionless time t/T are presented in Figs. 12–15 for a case with $ah/U^2 = 0.75$. The results for two, four, and eight elements are plotted against the exact solution available in Ref. 12. It can easily be seen that $N = 8$ gives acceptable results for all the states.

The control angle u vs dimensionless time t/T is presented in Fig. 16. Once again, the results are seen to be excellent for $N = 8$. In Table 1, the initial control $u(t_0)$ (which can be easily shown to be related to ν_1, ν_2, and ν_3) and the normalized final time aT/U are shown to converge quite rapidly as N is increased. Note, however, that the $N = 2$ and $N = 4$ approximations for $u(t_0)$ are neither upper nor lower bounds. This is a common characteristic of mixed formulations.

As with the fixed time problem, three of the four costates are constants. The costate results are all as accurate or better than the costate depicted in Fig. 10.

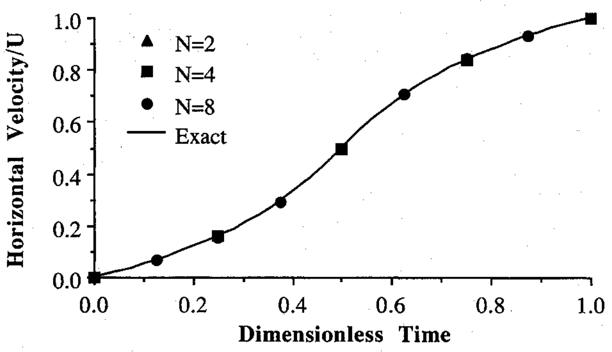

Fig. 14 Dimensionless horizontal velocity $x_{(3)}/U$ vs t/T. (The final horizontal component of velocity is constrained to be U.)

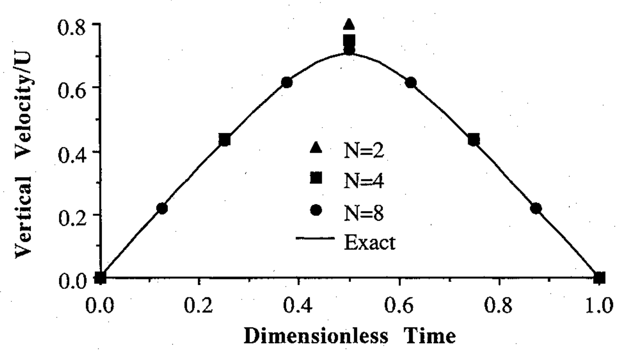

Fig. 15 Dimensionless vertical velocity $x_{(4)}/U$ vs t/T.

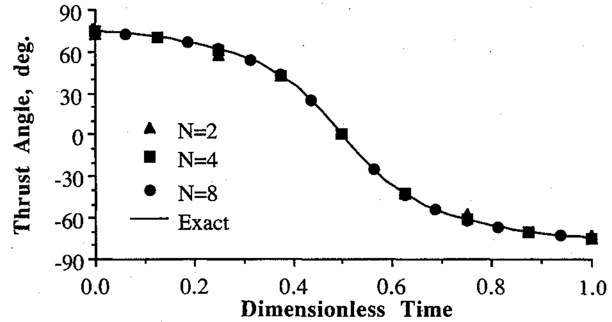

Fig. 16 Control angle u vs t/T.

Table 1 Convergence of $u(t_0)$ and aT/U vs N

N	$u(t_0)$, deg	aT/U
2	72.586	1.8819
4	74.736	1.8531
8	75.027	1.8413
16	74.969	1.8380
32	74.950	1.8372
Exact	74.944	1.8369

It should be noted that the computer time on a SUN 3/260 is only about 2 s for $N = 2$, $N = 4$, and $N = 8$ and 3 s for $N = 16$. Thus, the run time is relatively insensitive to N. These run times were achieved by using Harwell[16] subroutine MA28, thus taking advantage of the high percentage of zeros in the Jacobian matrix.

Conclusions and Future Work

In this paper, a mixed form of Hamilton's weak principle has been stated for dynamics problems. Finite elements in time were applied to this formulation and a simple initial value problem has been used to demonstrate the principles involved. A temporal finite element method based on a mixed form of the Hamiltonian weak principle was then developed for optimal control problems from the dynamics principles. It has been demonstrated that the mixed form allows for a simple choice of shape functions and is essentially self-starting. Two simple optimal control problems have been examined and the results are seen to provide excellent agreement with the exact solutions for even a very few elements. Overall, the method provides very accurate results for the problems investigated to date, with only a few elements and for minimal computational effort.

Future research will be done in applying this method to practical problems, such as development of onboard trajectory optimization algorithms for launch vehicles.[17] Such applications will require the reliability, efficiency, and self-starting characteristics illustrated in the present approach. The method will also have to be extended to allow for inequality constraints on the states and controls, as well as discontinuities in the states and state equations (such as from staging of a rocket).

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